A Bound for the Diameter of a Distance-regular Graph

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Abstract

In this MTH 501 project, we present the results of the article "The Diameter of Bipartite Distance-Regular Graphs" by Paul Terwilliger [1]. The work states and proves a conjectured upper bound for the diameter of a bipartite graph G based on the finite valency k and girth g. In this way, the results provide a (finite) upper bound on the number of distinct graphs with a given (finite) girth and valency. The proof techniques reveal the close interplay between graph theory and combinatorics.

A Math 501 Project

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Contents

1 Introduction

In this MTH 501 project, we present the results of the article "The Diameter of Bipartite Distance-Regular Graphs" by Paul Terwilliger [1]. The work states and proves a conjectured upper bound for the diameter of a bipartite graph G based on the finite valency k and girth g. In this way, the results provide a (finite) upper bound on the number of distinct graphs with a given (finite) girth and valency. The proof techniques reveal the close interplay between graph theory and combinatorics.

We begin by reviewing a well-known topic from classical geometry – the family of five shapes known as the platonic solids. The vertices and edges of these shapes form five familiar examples of distance-regular graphs, which are the topic of this project. Since the cube is bipartite, we will use these graphs of the platonic solids as running examples to which we repeatedly refer throughout this paper. By doing so, we are using familiar objects as a way to illustrate the main ideas and results of the paper.

1.1 History and Context

Since antiquity, mathematicians have been fascinated by the properties of the five platonic solids – the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron. These objects are pictured in Figure 1 below, and might be familiar to a reader as commonly used shapes for dice used in role-playing games like Dungeons & Dragons ©.

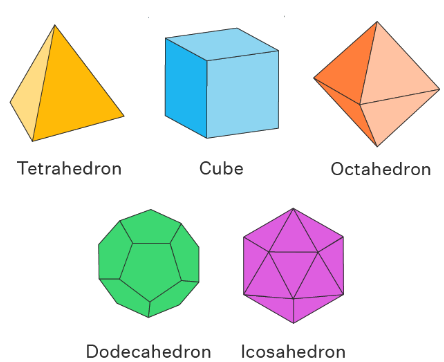

Figure 1: The five platonic solids (image: cuemath.com)

From a mathematical viewpoint, these five are the only convex solids that have congruent faces that are regular polygons. Their 1-skeletons consist of the vertices and edges and, as we will see in the next chapter, they can each be represented as planar graphs.

1.2 Organization of Topics

The content of this project is organized as follows. Chapter 1 contains a brief introduction to the topic and to the five platonic solids, in order to offer a bit of historical and mathematical context for this work. Chapter 2 presents the basic properties of graphs, with an emphasis on the terminology needed to state and prove the main results. In Chapter 3, we focus in on the main results, using combinatorial techniques to prove the diameter bound. Finally, in Chapter 4, we conclude with a number of observations and possible directions for further research.

2 Definitions

In this chapter, we begin with some basic definitions from graph theory required to present our work. For more background not included here, we refer the interested reader to the classic textbook by West [5].

2.1 Graphs

We begin with a few basic terms regarding graphs.

A **graph** X consists of a **vertex set** $V(X)$, an **edge set** $E(X)$, and a relation that associates with each edge two vertices called its **endpoints**. A graph is **simple** if it has no loops (an edge whose endpoints are equal) or multiple edges (edges whose pair of endpoints are the same). We can therefore specify a simple graph by its vertex set and its edge set, and we will be concerned only with finite undirected graphs. When u and v are the endpoints of an edge, they are said to be **adjacent**, written $u \sim v$. In this case, we refer to the shared edge as uv.

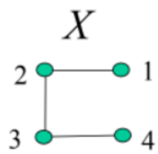

Figure 2: A graph with 4 vertices and 3 edges

By way of example, the simple graph in Figure 2 has four vertices and three edges. Notice that vertices 2 and 3 are adjacent, while vertices 1 and 4 are not. We will see many more examples of finite simple graphs in the work ahead.

2.2 Distance and Diameter

A **path** in a graph X is a sequence of distinct vertices $\{v_1, v_2, \ldots, v_n\}$ where $v_i \sim v_{i+1}$ for each i $(1 \leq i \leq n-1)$. The **length** of a path with n vertices is defined to be $n-1$. A graph is **connected** if, for any vertices $u, v \in V(X)$, there exists a path $\{v_1, v_2, \ldots, v_n\}$ where $v_1 = u$ and $v_n = v$. For any vertices $u, v \in V(X)$ let $\partial(u, v)$ denote the length of a shortest path from u to v. We refer to $\partial(u, v)$ as the **distance** between u and v. Then the **diameter** of X is the maximum value of $\partial(u, v)$ for any $u, v \in V(X)$.

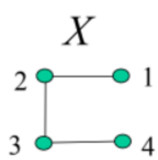

Figure 3: Vertices 1 and 3 are distance 2 apart

By way of example, the vertices 1 and 3 are distance two apart in the graph of Figure 3. That graph is connected and has diameter 3. In fact, the entire graph is a path from vertex 1 to vertex 4 of length 3.

As further examples of distance and diameter, we consider the platonic solids given in Figure 1. Here we find that the tetrahedron has diameter 1, the octahedron has diameter 2, the cube has diameter 3, the dodecahedron has diameter 5, and the icosahedron has diameter 3.

2.3 Cycles and Girth

A **cycle** in a graph X is a sequence of at least three distinct vertices $\{v_1, v_2, \ldots, v_n\}$ where where $v_i \sim v_{i+1}$ for each i $(1 \leq i \leq n-1)$ and where $v_n \sim v_1$. The **length** of a cycle with n vertices is defined to be n.

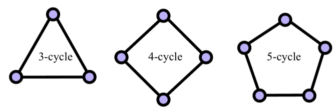

Figure 4: Cycles with 3, 4, and 5 vertices

By way of example, the graphs shown in Figure 4 represent cycles of length 3, 4, and 5. The **girth** of a graph X, denoted by $g = g(X)$, is the number of vertices in a shortest cycle in X.

As further examples of cycles and girth, we return to the platonic solids given in Figure 1. Here we find that the tetrahedron has girth 3, the octahedron has girth 3, the cube has girth 4, the dodecahedron has girth 5, and the icosahedron has girth 3.

2.4 Bipartite Graphs

A graph X is said to be **bipartite** if there are no cycles with an odd number of vertices. Equivalently, this means that there exists a partition of $V(X)$ into two nonempty sets A, B such that every edge of X has one endpoint in A and the other endpoint in B.

By way of example, the graph shown in Figure 5 is bipartite, with the indicated bipartition of the vertex set. Since each edge has one endpoint in the left cell of the bipartition and one endpoint in the right cell of the bipartition, any path must alternate between cells. This implies that all cycles have odd length.

As further examples, we return to the platonic solids given in Figure 1. Here we find that, since the tetrahedron, octahedron, dodecahedron, and the icosahedron have odd girth,

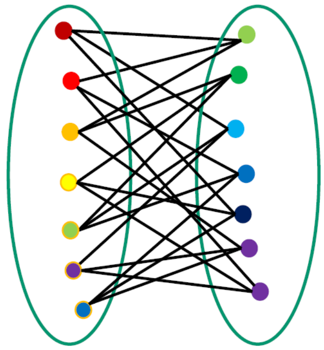

Figure 5: A bipartite graph with 14 vertices

none of them can be bipartite. On the other hand, the cube has girth 4, and in fact is easily seen to be bipartite.

2.5 Vertex Subgraph

If S is a nonempty subset of $V(X)$, we define the **vertex subgraph** X' **induced** by S to have vertex set $V(X') = S$ and edge set $E(X') = \{(u, v)|(u, v) \in E(X) \text{ and } u, v \in S\}$.

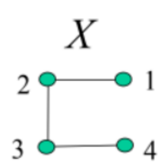

Figure 6: A graph with 4 vertices and 3 edges

By way of example, we again consider the graph shown in Figure 6. In this case, if we let $S = \{2, 3, 4\}$, then the subgraph induced by S is a path of length 2, Since X has 4 vertices, it has $2^4 - 1 = 15$ different non-trivial induced subgraphs.

2.6 Regular with Valency k

Suppose X is a connected graph. For any non-negative integers i and j and any vertices $u, v \in V(X)$, we denote by $P_{i,j}(u, v)$ the set of vertices a distance i from u and j from v.

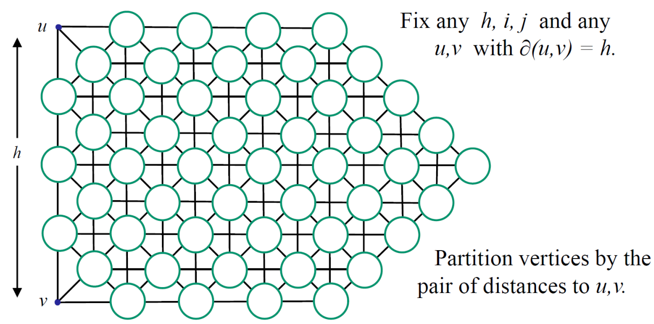

Figure 7: Partition vertices by distance from u and v

We say X is **regular, with valency** k whenever $P_{1,1}(u,u) = k$ for every vertex u. In other words, X is regular if every vertex has the same number of neighbors.

2.7 Distance-regular Graphs

A graph X with diameter d is said to be **distance-regular** whenever there exist integers p_{ij}^h $(0 \leq h, i, j \leq d)$ such that, for any vertices $u, v \in V(X)$ with $\partial(u,v) = h$,

$$p_{ij}^h = |\{w \in V(X) \,|\, \partial(u,w) = i \quad \text{and} \quad \partial(w,v) = j\}|.$$

The parameters p_{ij}^h are known as the **intersection numbers** of the graph X, and their definition is illustrated in Figure 8 below.

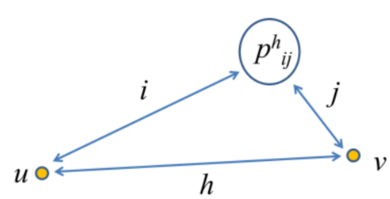

Figure 8: Defining distance-regularity

In referring to the parameters p_{ij}^h, it is common to restrict our attention to a smaller set of parameters from which the remainder can be easily deduced.

We let, for each integer i ($0 \le i \le d$):

$$c_i = p^i_{i-1,1} \qquad a_i = p^i_{i,1} \qquad b_i = p^i_{i+1,1}.$$

These count, respectively, the number of vertices z that are one step closer than, at the same distance as, and one step farther than vertex v from the vertex u. This smaller set of parameters is known as the **intersection array** of the graph and these parameters are illustrated in Figure 9 below.

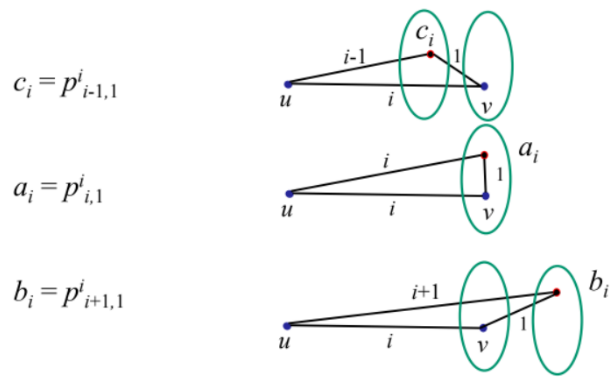

Figure 9: Intersection array parameters

To describe the parameters of a distance-regular graph, it is common to collect the values of the intersection array into a $3 \times (d+1)$ matrix as shown in Figure 10.

$$\left\{ \begin{array}{ccccccc} 0 & c_1 & c_2 & c_3 & \cdots & c_{d-1} & c_d \\ a_0 & a_1 & a_2 & a_3 & \cdots & a_{d-1} & a_d \\ b_0 & b_1 & b_2 & b_3 & \cdots & b_{d-1} & 0 \end{array} \right\}$$

Figure 10: Intersection array for diameter d

By way of example, we return to the platonic solids given in Figure 1. For each, we give the intersection array of the corresponding distance-regular graph.

Intersection Array for Tetrahedron

$$\left\{\begin{array}{cc} 0 & 1 \\ 0 & 2 \\ 3 & 0 \end{array}\right\}$$

Intersection Array for Octahedron

$$\left\{\begin{array}{ccc} 0 & c_1 & c_2 \\ a_0 & a_1 & a_2 \\ b_0 & b_1 & 0 \end{array}\right\}$$

Intersection Array for Cube

$$\left\{\begin{array}{cccc} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right\}$$

Intersection Array for Dodecahedron

$$\left\{\begin{array}{cccccc} 0 & c_1 & c_2 & c_3 & c_4 & c_5 \\ a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ b_0 & b_1 & b_2 & b_3 & b_4 & 0 \end{array}\right\}$$

Intersection Array for Icosahedron

$$\left\{\begin{array}{cccc} 0 & c_1 & c_2 & c_3 \\ a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & 0 \end{array}\right\}$$

3 Main Results

In this chapter we present the proof of the main result of the paper by Paul Terwilliger [1]. We begin with a bit of motivation for the result.

3.1 Motivation

A long-standing open question concerns whether or not there are finitely many distance-regular graphs with a given valency k. Since the 1970s, mathematicians have studied the special case of distance-transitive graphs. For example, Biggs and Smith found the complete list of distance-transitive graphs with valency 3 or 4. A result of Weiss proved that the girth of any finite distance-transitive graph is at most 16, under mild assumptions. The distance-regular case, however, remained open for years. In this section we prove the Terwilliger theorem that implies the result for the case when the graph is bipartite and distance-regular.

3.2 Statement of Main Result and Set-Up

Theorem 1 *Let X be any bipartite, distance-regular graph with intersection array*

$$\{1 = c_1, c_2, \ldots, c_i, \ldots; k, b_1, b_2, \ldots, b_i, \ldots\}. \tag{1}$$

Then for any positive integer n, less than or equal to the diameter of X, $c_h > 1$ implies there exists an i $(1 \leq i \leq h-1)$ where $c_h \geq c_i + c_{h-i}$.

Remark 1 *(The subgraph B)* Before proving the result above, we introduce an important subgraph to which we will often refer. Let u and v be any vertices of a bipartite, distance-regular graph X. Suppose $\partial(u, v) = h$ and let B denote the subgraph induced on S, where

$$S = \{x \in V(X) \mid \partial(u, x) + \partial(x, v) = h\}.$$

Lemma 1 *If x is any vertex in B (defined above) and if $\partial(x, u) = i$ $(1 \leq i \leq h-1)$, then the valency of x in B is $c_i + c_{h-1}$. The valency of u and v in B is c_h.*

Proof. This is immediate from the fact that the set of vertices in B adjacent to x is $P_{i-1,1}(u,x) \cup P_{1,h-i-1}(x,v)$, and the set of vertices in B adjacent to u and v are $P_{1,h-1}(u,v)$ and $P_{h-1,1}(u,v)$, respectively. $\square$

Lemma 2 *For any pair of vertices a and b in B (defined above),*

$$\partial(a,b) \leq \partial(u,v) = h$$

where distances are measured in X.

Proof. From the definition of B, we have

$$\partial(a,v) + \partial(a,u) = h, \tag{2}$$

$$\partial(b,v) + \partial(b,u) = h. \tag{3}$$

Combining (2) and (3) we get

$$\partial(a,u) + \partial(b,u) + \partial(a,v) + \partial(b,v) = 2h.$$

Therefore either

$$\partial(a,u) + \partial(b,u) \leq h$$

or

$$\partial(a,v) + \partial(b,v) \leq h.$$

Without loss of generality, suppose that

$$\partial(a,u) + \partial(b,u) \leq h.$$

Then

$$\partial(a,b) \leq \partial(a,u) + \partial(u,b) \leq h,$$

as required. $\square$

3.3 Proof of Main Theorem

Now we are ready to prove Theorem 1, as stated earlier. We restate the theorem here for convenience.

Theorem 1 *Let X be any bipartite, distance-regular graph with intersection array*

$$\{1 = c_1, c_2, \ldots, c_i, \ldots; k, b_1, b_2, \ldots, b_i, \ldots\}.$$

Then for any positive integer h less than or equal to the diameter of X, if $c_h > 1$, then there exists an i $(1 < i < h-1)$ where $c_h \geq c_i + c_{h-i}$.

Proof. By way of contradiction, assume that for some positive integer h and for all i $(1 \leq i \leq h-1)$ we have $c_i + c_{h-i} > c_h$. We will show that $c_h = 1$. Let u and v be vertices in X satisfying $\partial(u, v) = h$. Let B be the subgraph of X defined at the beginning of this section.

We now define a function $f : V(B) \times V(B) \to \mathbb{Z}$, where $\mathbb{Z}$ denotes the integers, by

$$f(b_1, b_2) = \partial(b_1, b_2) - |\partial(b_1, u) - \partial(b_2, u)|, \quad b_1, b_2 \in V(B),$$

where the vertical bars denote absolute value. Note that $f(u, v) = 0$. Using Lemma 2 and the triangle inequality we can conclude that

$$0 \leq f(b_1, b_2) \leq h, \quad b_1, b_2 \in V(B).$$

Let

$$f_{max} = \max\{f(b_1, b_2) | b_1, b_2 \in V(B)\}.$$

Claim. We claim $f_{max} = 0$, and hence $f(b_1, b_2) = 0$ for all $b_1, b_2 \in V(B)$.

Proof of Claim. Let $S \subseteq V(B) \times V(B)$ be the set of pairs (b_1, b_2) of vertices in B where $f(b_1, b_2) = f_{max}$. Out of the set S, choose a pair (b_1, b_2) where

$$|\partial(b_1, u) - \partial(b_2, u)|$$

is a maximum. We show that $(b_1, b_2) = (u, v)$ or (v, u) and hence $f_{max} = f(u, v) = 0$. Suppose $(b_1, b_2) \neq (u, v)$ or (v, u). We can assume without loss of generality that $a \neq u$ or v. Let

$$\partial(b_1, u) = i, \quad 1 \leq i \leq h - 1.$$

From Lemma 1, the valency of b_1 in B is then $c_i + c_{1-i}$, which we are assuming is strictly greater than c_h. From Lemma 2, we must have $\partial(b_1, b_2) \leq h$, where the distance is measured in Γ. Let $\partial(b_1, b_2) = \ell$ for some ℓ, where $\ell \leq h$. Since Γ is distance-regular, there are exactly c_ℓ vertices in Γ adjacent to b_1 and closer to b_2 than b_1. Since Γ is bipartite the remaining $k - c_\ell$ vertices adjacent to b_1 are further from b_2 than b_1 is. There are more than c_h vertices in B that are adjacent to b_1. Since $\ell \leq h$, we have $c_h \geq c_\ell$, so there must be at least one vertex b_1' in B, adjacent to b_1, but further from b_2 than b_1 is. That is,

$$\partial(b_1', b_2) = \partial(b_1, b_2) + 1 = \ell + 1.$$

We now calculate $f(b_1', b_2)$. We have

$$\begin{aligned} f(b_1', b_2) &= \partial(b_1', b_2) - |\partial(b_1', u)\partial(b_2, u)| && (4) \\ &= \partial(b_1, b_2) + 1 - |\partial(b_1', u - \partial(b_2, u)|. && (5) \end{aligned}$$

Since $f(b_1, b_2) = f_{max}$, the maximum value for f, we must have

$$f(b_1', b_2) \leq f(b_1, b_2). \tag{6}$$

Equations (3), (4), and (5) imply

$$|\partial(b_1', u) - \partial(b_2, u)| \geq |\partial(b_1, u) - \partial(b_2, u)| + 1. \tag{7}$$

But, since b_1' is adjacent to b_1, we also have

$$|\partial(b_1', u) - \partial(b_2, u)| \leq |\partial(b_1, u) - \partial(b_2, u)| + 1,$$

so we must have equality. Therefore

$$f(b_1, b_2) = f(b_1', b_2)$$

and so

$$(b_1', b_2) \in S.$$

But now (6) and (7) imply that the pair (b_1, b_2) was not the element in S where

$$|\partial(b_1, u) - \partial(b_2, u)|$$

was a maximum. Therefore our assumption that $b_1 \neq u$ or v must be wrong.

By a similar argument, we can show that $b_2 = u$ or v as well, and thus $(b_1, b_2) = (u, v)$ or (v, u). Hence, $f_{max} = f(u, v) = 0$ and the claim is proved. $\square$

With this claim, we can now complete the proof. If c_h were not 1, then $P_{1,h-1}(u, v)$ would contain at least two distinct vertices denoted by x, y. Then we would have

$$\begin{aligned} f(x, y) &= \partial(x, y) - |\partial(x, u) - \partial(y, u)| \\ &= \partial(x, y) - 0 \\ &> 0. \end{aligned}$$

Since $f(x, y)$ was shown to be 0 for all $x, y \in V(B)$, we have a contradiction. Thus, we must conclude that, under the assumptions of this proof, the value of $c_h = 1$. $\square$

3.4 Main Takeaway

Corollary 1 *Let X be any connected bipartite distance-regular graph with at least one cycle. Then X is finite, with diameter d, valency k and girth g satisfying*

$$d \leq \frac{(k-1)(g-2)}{2} + 1.$$

Proof. Let

$$\{1, c_2, \ldots, c_i, \ldots; k, b_1, \ldots, b_i, \ldots\}$$

be the intersection array for X. Since X has a cycle, X has finite girth which will denote by g. Since X is bipartite, g is even, and $c_{g/2} > 1$. We claim that

$$c_i \geq \frac{2i}{g-2}, \quad i \geq 1. \tag{8}$$

The proof is by induction on i. Clearly, if $1 \leq i < \frac{g}{2}$, we have $c_i = 1$, which certainly implies $c_i \geq \frac{2i}{g-2}$. Now pick any $i \geq \frac{g}{2}$ and by induction assume that

$$c_\ell \geq \frac{2\ell}{g-2} \quad \text{for all} \quad \ell, \quad 1 \leq \ell < i. \tag{9}$$

Since $i \geq \frac{g}{2}$ we have $c_i > 1$, so by Theorem 1, there must be a positive integer j, such that $1 \leq j \leq i-1$, with

$$c_j + c_{i-j} \leq c_i.$$

From (9)

$$c_j \geq \frac{2j}{g-2}$$

and

$$c_{i-j} \geq \frac{2(i-j)}{g-2}.$$

This gives

$$c_i \geq c_j + c_{i-j} \geq \frac{2j}{g-2} + \frac{2(i-j)}{g-2} = \frac{2i}{g-2}$$

as required. Denoting the diameter by d and using (8) we get

$$k - 1 \geq c_{d-1} \geq \frac{2(d-1)}{g-2}$$

or

$$d \leq \frac{(k-1)(g-2)}{2} + 1$$

as desired. $\square$

Corollary 2 *For any integers k and g where $k, g > 2$, there are only finitely many bipartite distance-regular graphs with valency k and girth g.*

Proof. Let X be any bipartite distance-regular graph with valency k and girth g. Fix any vertex x in X. Partition the vertices of X according to their distance from x, so that

$$V(X) = V_0 \cup V_1 \cup \cdots \cup V_d,$$

where each V_i is the set of vertices at distance i from x and where the diameter d is at most $(k-1)(g-2)/2+1$ by the theorem above. Since the valency is k, we have $|V_i| \leq k(k-1)^{i-1}$ for each $i > 0$. Therefore.

$$|V(X)| \leq 1 + k + k(k-1) + k(k-1)^2 + \cdots + k(k-1)^{d-1},$$

which is finite. If N denotes the value on the right side of the expression above, then X has at most N vertices. There are only finitely many graphs on N vertices, hence there are only finitely many choices for X. $\square$

3.5 Applications to Hypercubes

We conclude with a number of examples to illustrate the power of the results and compare the upper bound given with the actual diameters of some graphs in this family. We begin with a simple example, the 2-dimensional hypercube, which is a 4-cycle.

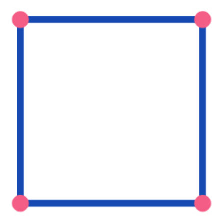

Figure 11: 2-dimensional hypercube

Example 1 Consider the hypercube in 2 dimensions. Then $k = 2$ and $g = 4$, so

$$\begin{aligned} d &\leq \frac{(2-1)(4-2)}{2} + 1 \\ &= \frac{(1)(2)}{2} + 1 \\ &= 1 + 1 \\ &= 2. \end{aligned}$$

In fact the diameter equals 2 for a hypercube in two dimensions. □

Example 2 Consider the hypercube in 3 dimensions. Then $k = 3$ and $g = 4$, so

$$\begin{aligned} d &\leq \frac{(3-1)(4-2)}{2} + 1 \\ &= \frac{(2)(2)}{2} + 1 \\ &= 2 + 1 \\ &= 3. \end{aligned}$$

In fact the diameter equals 3 for a hypercube in three dimensions. $\square$

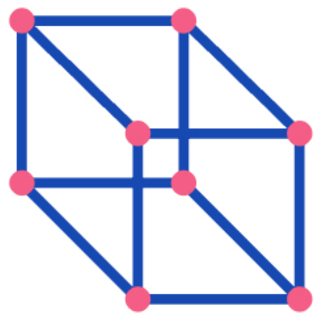

Figure 12: 3-dimensional hypercube

Example 3 Consider the hypercube in 4 dimensions. Then $k = 4$ and $g = 4$, so

$$\begin{aligned} d &\leq \frac{(4-1)(4-2)}{2} + 1 \\ &= \frac{(3)(2)}{2} + 1 \\ &= 3 + 1 \\ &= 4. \end{aligned}$$

In fact the diameter equals 4 for a hypercube in four dimensions. $\square$

Remark 2 Note that for the family of hypercubes, we have equality for d and its upper bound. Moreover, $d = k$, the valency. Coincidentally, for the family of hypercubes in n-dimensions, the valency k will be equal to n. Thus, the diameter and valency are both equal to the dimension of the hypercube.

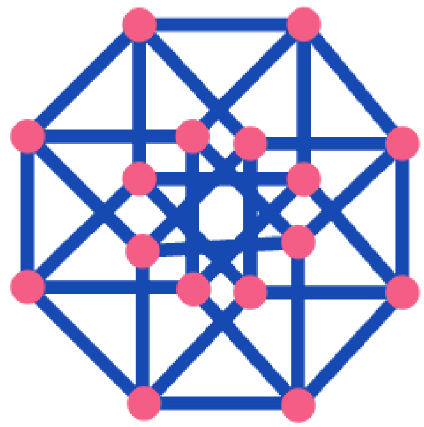

Figure 13: 4-dimensional hypercube

4 Addendum on Archimedean Solids

As a counterpoint, we consider the case of the Archimedean solids. These are shown in Figure 14, and we can note that the truncated octahedron, the great rhombicuboctahedron, and the great rhombicosidodecahedron are bipartite (since every face is even).

Moreover, the valency for each of these bipartite shapes is $k = 3$ and the girth is $g = 4$ (as the smallest cycle in each shape is a 4-cycle). However, these shapes are not distance-regular. Hence, these shapes do not meet the upper bound mentioned in this paper.

Example 4 Consider the truncated octahedron. Then observe that $k = 3$ and $g = 4$, so

$$\begin{aligned} d &\leq \frac{(3-1)(4-2)}{2} + 1 \\ &= \frac{(2)(2)}{2} + 1 \\ &= 2 + 1 \\ &= 3. \end{aligned}$$

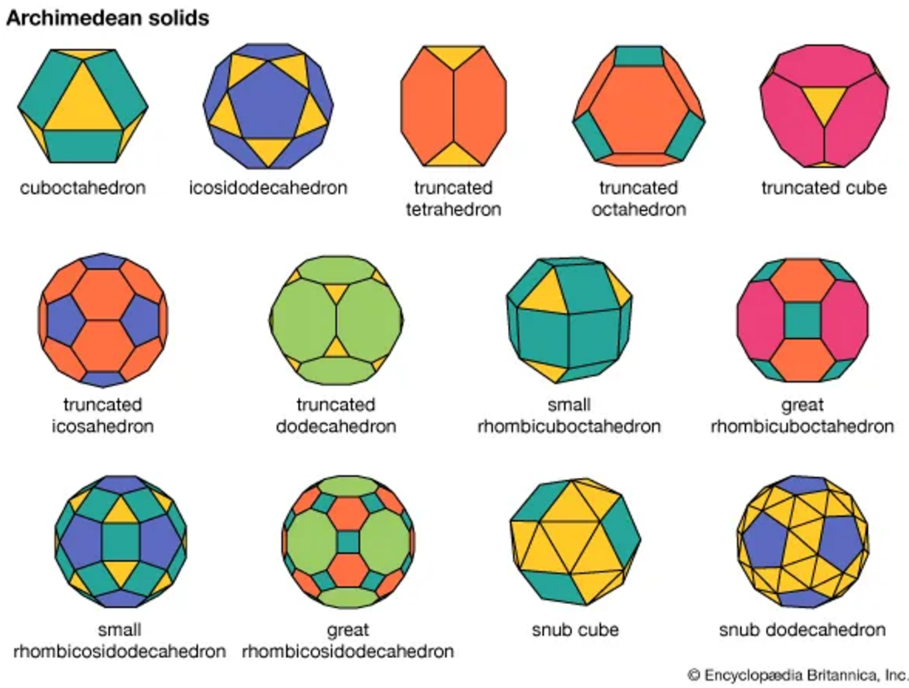

Figure 14: Archimedean Solids

However, the diameter of the truncated octahedron is $d = 6$.

Example 5 Consider the great rhombicuboctahedron. Then observe that $k = 3$ and $g = 4$, so

$$\begin{aligned} d &\leq \frac{(3-1)(4-2)}{2} + 1 \\ &= \frac{(2)(2)}{2} + 1 \\ &= 2 + 1 \\ &= 3. \end{aligned}$$

In fact the diameter for the great rhombicuboctahedron is $d = 9$. □

Example 6 Consider the great rhombicosidodecahedron. Then observe that $k = 3$ and

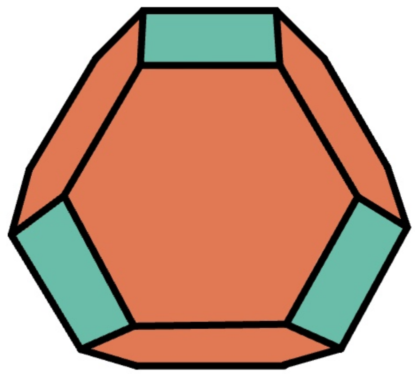

Figure 15: Truncated Octahedron

$g = 4$, so

$$\begin{aligned} d &\leq \frac{(3-1)(4-2)}{2} + 1 \\ &= \frac{(2)(2)}{2} + 1 \\ &= 2 + 1 \\ &= 3. \end{aligned}$$

However, the diameter of the great rhombicosidodecahedron is $d = 12$.

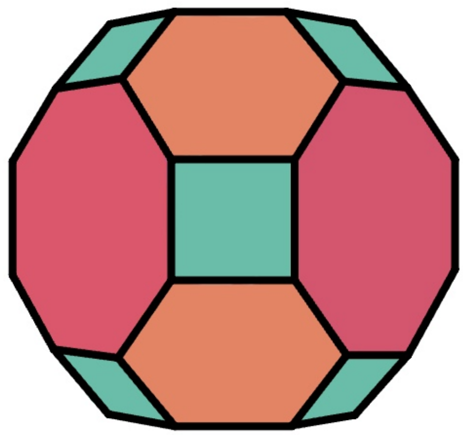

Figure 16: Great Rhombicuboctahedron

5 Conclusion

In this paper, we have reviewed, expanded, and applied the results in the work of Paul Terwilliger on the upper bound for the diameter of bipartite, distance-regular graphs.

Of particular interest to me was the interplay of platonic solids, polygons, combinatorics, and graphs. My hope was to deepen my understanding of the dual polyhedrons (based off of the platonic solids) by looking at the platonic solids and their properties in the realm of graph theory. Since my first exposure to such questions stemmed from combinatorial counting arguments, it made sense to find a topic that had the best of both worlds. Hence, this project has taken on its overall focus.

Further results beyond the scope of this project include providing a formal argument that the family of hypercubes has $d = k = n$ in n-dimensional spaces for all $n \in \mathbb{N}$.

Another interesting thing to pursue further might be to explore applications to dual polyhedrons and Johnson solids.

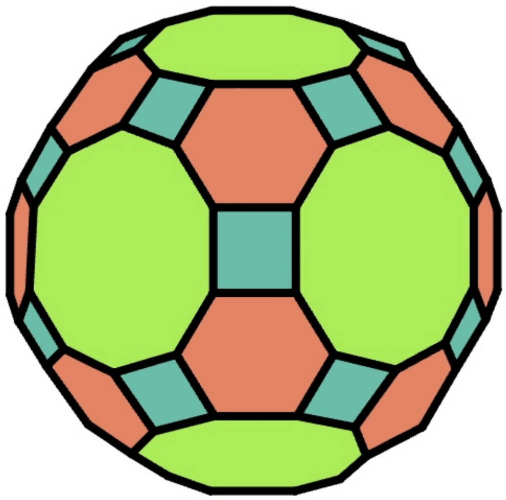

Figure 17: Great Rhombicosidodecahedron

For more information on these topics and others, the motivated reader may be interested in exploring the resources listed in the references below.

References

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